TRANSFORMATIONS PRESERVING THE GRASSMANNIAN

BY
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1. **Introduction.** For m a positive integer, let E_m be the arithmetic m-space over a commutative field F. Let $\mathscr{A}_m$ be the full linear group of E_m, and let S_{m-1} be the projective space of homogeneous coordinates in E_m. For the rest of the paper, we fix two positive integers n and k, such that $k < n$. Let $N = \binom{n}{k}$, and let $\Omega(k,n)$ be the k,n Grassmannian variety:

$$\Omega(k,n) \subset S_{N-1}.$$

Let $\psi(k,n)$ be the set of those nonzero elements x of E_N such that there is some y satisfying

$$x \in y \in \Omega(k,n).$$

Let G be the set of nonsingular linear transformations of E_N which keep $\psi(k,n)$ fixed as a set. If C_N is the center of the full linear group of E_N, then G/C_N is the set of projective transformations of S_{N-1} which keep $\Omega(k,n)$ fixed as a set.

Let $A(n,k)$ be the group of all k-compounds [1, Vol. 1, p. 291] of elements of $\mathscr{A}_n$. Then $A(n,k)/(C_N \cap A(n,k))$ may be thought of as the group of projective transformations of S_{N-1} "induced" by the group of projective transformations of S_{n-1}. Since $A(n,k)/(C_N \cap A(n,k))$ is isomorphic to $(A(n,k)\cdot C_N)/C_N$, and since $A(n,k)\cdot C_N$ is a subgroup of G, $(A(n,k)\cdot C_N)/C_N$ is a subgroup of G/C_N.

The principal results to be proved here are:

1. If $n \neq 2k$, then

$$A(n,k)\cdot C_N = G,$$

and thus

$$(A(n,k)\cdot C_N)/C_N = G/C_N.$$

2. If $n = 2k$, let J denote the "star dual" mapping of $\psi(k,n)$ onto itself (see 2). Since

$$J^2 = (-1)^{(k^2)} I,$$

where I is the identity element of $\mathscr{A}_N$, J generates a cyclic subgroup of order 2 if k is even, and of order 4 if k is odd. Let $\mathscr{J}$ denote this group. Let $\mathscr{K}$ be the

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subgroup of G/C_N made up of cosets of elements of $\mathscr{J}$. Thus $\mathscr{K}$ is of order 2. Then, in this case,

$$\mathscr{J} \cdot A(n,k) \cdot C_N = G,$$

and thus

$$\mathscr{K} \cdot ((A(n,k) \cdot C_N)/C_N) = G/C_N.$$

2. **Notation.** (For definitions of terms used here and proofs of results given here, see [2].) We shall denote the exterior product of vectors by "$\wedge$". Thus x is an element of $\psi(k,n)$ if and only if there is a linearly independent set of k elements of $E_n, x_1, x_2, x_3, \cdots, x_k$; and

$$x = x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_k.$$

For $A \in \mathscr{A}_n$, let A^k be the k-compound of A. Thus if

$$x = x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_k,$$

then

$$A^k x = Ax_1 \wedge Ax_2 \wedge Ax_3 \wedge \cdots \wedge Ax_k.$$

For $E \subset E_m$, let $L(E)$ be the subspace of E_m spanned by E. If $x \in \psi(k,n)$, such that

$$x = x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_k,$$

let

$$\pi(x) = L(\{x_1, x_2, x_3, \cdots, x_k\}).$$

For any positive integer m, let

$$\mathscr{N}(m) = \{1, 2, 3, \cdots, m\}.$$

For t a positive integer, $t \leqq m$, let

$$P(m,t) = \{p : p = \{p_1, p_2, p_3, \cdots, p_t\},\ p_i \in \mathscr{N}(m) \text{ for } i \in \mathscr{N}(t), \text{ and}$$
$$p_1 < p_2 < p_3 < \cdots < p_t\}.$$

For $p \in P(m,t)$, let $c(p)$ be that element of $P(m, m-t)$ such that

$$p \cup c(p) = \mathscr{N}(m).$$

For x an element of $\psi(k,n)$, $*x$ is that element of $\psi(n-k,n)$ defined by

$$(*x)_q = \varepsilon(q)\, x_{c(q)},$$

where q is any element of $P(n, n-k)$, and $\varepsilon(q)$ is -1 to the power of the parity of the permutation $(q_1, q_2, q_3, \cdots, q_{n-k}, (cq)_1, (cq)_2, (cq)_3, \cdots, (cq)_k)$. Let J be that mapping of $\psi(k,n)$ onto $\psi(n-k,n)$ defined by

$$J(x) = *x.$$

Then J can be extended to a nonsingular linear mapping of E_N onto itself.

Since $k < n$, we may consider E_{k+1} as a subspace of E_n, and $\psi(k, k+1)$ as a subset of $\psi(k,n)$. On occasion, we shall find it necessary to use the $*$-dual of a vector in $\psi(k,k+1)$ "relative to E_{k+1}." That is, for x an element of $\psi(k,k+1) \subset \psi(k,n)$,

$$(*_{k+1}x)_i = (-1)^{i-1}x_{c(i)}, \quad \text{where } c(i) = \mathcal{N}(k+1) - \{i\}, \text{ if } 1 \leqq i \leqq k+1;$$

and

$$(*_{k+1}x)_i = 0, \qquad \text{if } i > k+1.$$

Then $*_{k+1}x \in E_{k+1} \subset E_n$, and

$$L(*_{k+1}x) = (\pi(x))^{\perp_{k+1}},$$

where $\perp_{k+1}$ denotes the orthogonal complement relative to E_{k+1}.

For $i \in \mathcal{N}(m)$, let e_i be that element of E_m whose jth component is δ_{ij}. For $p \in P(n,k)$, let

$$e_p = e_{p_1} \wedge e_{p_2} \wedge e_{p_3} \wedge \cdots \wedge e_{p_k}.$$

Then the set $\{e_p\colon\ p \in P(n,k)\}$ is a basis for E_N.

For $A \in G$, and $p \in P(n,k)$, let $A_p = Ae_p$. Then $A_p \in E_N$, and it is the pth column vector of the matrix of A. For any $q \in P(n,k-1)$,

$$\dim\left(\bigcap \pi(e_p)\right) = k-1,$$

the intersection being taken over all $p \in P(n,k)$ such that $q \subset p$; and

$$\dim(L(\{e_p : q \subset p \in P(n,k)\})) = n-k+1.$$

So if $A \in G$, and $q \in P(n,k-1)$, and if

$$M = A(L(\{e_p\colon\ q \subset p \in P(n,k)\})),$$

then $\dim M = n-k+1$, and M is spanned by the set $\{A_p\colon\ q \subset p \in P(n,k)\}$. Furthermore, for $p \in P(n,k)$, $A_p \in M$ if and only if $q \subset p$.

Since we have excluded the zero vector from $\psi(k,n)$, no linear subspace of E_N is contained in $\psi(k,n)$. However, if M is a linear subspace of E_N, we shall say $M \subset \psi(k,n)$ if and only if for $x \in M$, if $x \neq 0$, then $x \in \psi(k,n)$.

3. **Principal results.** The principal results may now be stated in the following two theorems.

3.1. THEOREM. *If $n \neq 2k$, and $A \in G$, then there exists $C \in C_N$ and $B \in \mathcal{A}_m$ such that*

$$A = CB^k.$$

3.2. THEOREM. *If $n = 2k$, and if $A \in G$, then there exists $C \in C_N$ and $B \in \mathcal{A}_n$ such that either*

$$A = CB^k$$

or

$$A = CJB^k.$$

The proofs of these theorems depend on the following three lemmas, which will be proved in §§4 and 5.

3.3. LEMMA. *For m an integer, $2 \leqq m \leqq N$, let M be a subspace of E_N, with $\dim M = m$, such that there exists a set $\{x_1, x_2, x_3, \cdots, x_m\} \subset \psi(k,n)$ and $\{x_1, x_2, x_3, \cdots, x_m\}$ spans M. Then,*

1. *if*

$$\dim \bigcap_{i=1}^{m} \pi(x_i) = k - 1,$$

then $M \subset \psi(k,n)$,

$$\dim \bigcap_{x \in M} \pi(x) = k - 1,$$

and

$$\dim L(\{\pi(x)\colon\ x \in M\}) = k + m - 1;$$

2. *if*

$$\dim L(\{\pi(x_i)\colon\ 1 \leqq i \leqq m\}) = k + 1,$$

then $M \subset \psi(k,n)$, $m \leqq k + 1$,

$$\dim \bigcap_{x \in M} \pi(x) = k - m + 1,$$

and

$$\dim L(\{\pi(x)\colon\ x \in M\}) = k + 1.$$

In either case, M is the set of all k-vectors of k dimensional subspaces of E_n which contain $\bigcap_{x \in M} \pi(x)$ and are contained in $L(\{\pi(x)\colon\ x \in M\})$.

3.4. LEMMA. *For m an integer, $2 \leqq m \leqq N$, let M be a subspace of E_N, with $\dim M = m$, and assume that $M \subset \psi(k,n)$. Let $\{x_1, x_2, x_3, \cdots, x_m\}$ be any spanning set of M. Then either*

$$\dim \bigcap_{i=1}^{m} \pi(x_i) = k - 1,$$

or

$$\dim L(\{\pi(x_i)\colon\ 1 \leqq i \leqq m\}) = k + 1.$$

3.5. LEMMA. *If $A \in G$, and if, for each $q \in P(n, k-1)$,*

$$\dim \bigcap \pi(A_p) = k - 1,$$

the intersection being taken over all p such that

$$q \subset p \in P(n,k),$$

then there exists $C \in C_N$ *and* $B \in \mathscr{A}_n$ *such that*

$$A = CB^k.$$

Proof of Theorem 3.1 assuming Lemmas 3.3, 3.4, and 3.5. First assume that $n > 2k$. For $q \in P(n,k-1)$, let $M(q)$ be the subspace of E_N spanned by the set $\{A_p\colon\ q \subset p \in P(n,k)\}$. Then $M(q) \subset \psi(k,n)$, and $\dim M(q) = n - k + 1$. But $n - k + 1 > k + 1$. So by 3.3 and 3.4,

$$\dim \bigcap \pi(A_p) = k - 1,$$

the intersection being taken over all p such that

$$q \subset p \in P(n,k).$$

The result follows from 3.5. Now assume that $n < 2k$. Then for $x \in \psi(n-k,n)$, $JAJ^{-1}(x) \in \psi(n-k,n)$. Hence there exists $C \in C_N$ and $B \in \mathscr{A}_n$ such that

$$JAJ^{-1} = CB^{n-k}.$$

So

$$A = CJ^{-1}B^{n-k}J.$$

By the Laplace expansion of a determinant,

$$J^{-1}B^{n-k}J = (\det B)\, I\,(B^{-T})^k,$$

where $-T$ denotes inverse transpose. Hence

$$A = C(\det B)\, I\,(B^{-T})^k.$$

This completes the proof.

Proof of Theorem 3.2 assuming Lemmas 3.3, 3.4, and 3.5. We first show that if

$$\dim L(\{\pi(A_p)\colon\ q' \subset p \in P(n,k)\}) = k + 1,$$

for some $q' \in P(n,k-1)$, then

$$\dim L(\{\pi(A_p)\colon\ q \subset p \in P(n,k)\}) = k + 1,$$

for every $q \in P(n,k-1)$. It suffices to consider $q' = \{1,2,3,\cdots,k-1\}$ and to assume that

$$\dim L(\{\pi(A_p)\colon\ q' \subset p \in P(n,k)\}) = k + 1.$$

Select $q \in P(n,k-1)$, so ordered that if $q_i \in q'$, then $q_i = i$. Let $q'' = \{2,3,4,\cdots,k-1,q_1\}$. We will show that

$$\dim L(\{\pi(A_p)\colon\ q'' \subset p \in P(n,k)\}) = k + 1.$$

If $q_1 = 1$, there is nothing to prove. So assume that $q_1 \neq 1$. Let $p'' = \{1,2,3,\cdots,k-1,q_1\}$, and let

$$M' = L(\{A_p\colon\ q' \subset p \in P(n,k)\}),$$

and

$$M'' = L(\{A_p\colon\ q'' \subset p \in P(n,k)\}).$$

Then

$$M' \cap M'' = L(A_{p''}),$$

so

$$\dim(M' \cap M'') = 1.$$

Now let $Q' = L(\{\pi(A_p)\colon\ q' \subset p \in P(n,k)\})$, and $Q'' = \bigcap \pi(A_p)$, the intersection being taken over all $p \in P(n,k)$ such that $q'' \subset p$, and assume that $\dim Q'' = k-1$. Then

$$Q'' \subset \pi(A_{p''}) \subset Q'.$$

So the set of all $y \in \psi(k,n)$ such that $Q'' \subset \pi(y) \subset Q'$ is a subspace of $M' \cap M''$, but by [1, Vol. 2, Chapter XIV, Theorem I], the dimension of this subspace is 2. So $\dim(M' \cap M'') \geqq 2$. This is a contradiction. So by Lemma 3.4,

$$\dim L(\{\pi(A_p)\colon\ q'' \subset p \in P(n,k)\}) = k+1.$$

Continuing in this manner, working with one element of q at a time, we conclude that

$$\dim L(\{\pi(A_p)\colon\ q \subset p \in P(n,k)\}) = k+1.$$

Hence either A or JA satisfies the conditions of Lemma 3.5, so the result follows from the fact that $J^2 = (-1)^{(k^2)}I$.

4. **Linear subspaces contained in $\psi(k, n)$.** Lemmas 3.3 and 3.4 describe the linear subspaces of E_N which are contained in $\psi(k,n)$ in the sense of 2. In this section we give proofs of these two lemmas.

Proof of Lemma 3.3. Select a set $\{x_1,x_2,x_3,\cdots,x_m\} \subset \psi(k,n)$, such that $\{x_1,x_2,x_3,\cdots,x_m\}$ spans M, and assume that

$$\dim \bigcap_{i=1}^{m} \pi(x_i) = k-1.$$

Then without loss of generality, we may assume that

$$x_i = e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_{k-1} \wedge e_{k+i-1}, \qquad \text{for } i = 1,2,3,\cdots,m.$$

Now let $x \in M$. Then there exist $a_1,a_2,a_3,\cdots,a_m$, elements of F, such that $x = \sum_{i=1}^{m} a_i x_i$. So

$$x = e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_{k-1} \wedge \left(\sum_{i=1}^{m} a_i e_{k+i-1}\right).$$

Hence $M \subset \psi(k,n)$, and consists of those k-vectors of k-spaces containing $L(\{e_1, e_2, e_3, \cdots, e_{k-1}\})$, and contained in $L(\{e_1, e_2, e_3, \cdots, e_{k+m-1}\})$. Now assume that

$$\dim L(\{\pi(x_i)\colon 1 \leqq i \leqq m\}) = k+1.$$

Then without loss of generality, we may assume that

$$\pi(x_i) \subset L(\{e_1, e_2, e_3, \cdots, e_{k+1}\})$$

for $i = 1, 2, 3, \cdots, m$. Hence the x_i may be thought of as k-vectors in E_{k+1}. So if $x \in M$, $x = \sum_{i=1}^{m} a_i x_i$, for suitable elements a_i of F, then x is a k-vector in E_{k+1}. Hence $M \subset \psi(k,n)$, and

$$\dim L(\{\pi(x)\colon x \in M\}) = k+1.$$

Also, the set $\{*_{k+1} x_i\colon 1 \leqq i \leqq m\}$ spans an m-space of E_{k+1}, so $m \leqq k+1$, and since $L(*_{k+1} x_i) = (\pi(x_i))^{\perp_{k+1}}$,

$$\dim \bigcap_{i=1}^{m} \pi(x_i) = k - m + 1.$$

But for $x \in M$, $L(*_{k+1} x) \subset L(\{*_{k+1} x_i\colon 1 \leqq i \leqq m\})$, and so

$$\bigcap_{i=1}^{m} \pi(x_i) \subset \pi(x).$$

Hence

$$\dim \bigcap \pi(x) = k - m + 1,$$

the intersection being taken over all $x \in M$. This completes the proof.

Proof of Lemma 3.4. Since $M \subset \psi(k,n)$, the plane spanned by x_i and x_j lies in $\psi(k,n)$, for $i \neq j$, $i,j = 1, 2, 3, \cdots, m$. By [1, Vol. 2, Chapter XIV, Theorem I],

$$\dim(\pi(x_i) \cap \pi(x_j)) = k-1.$$

So, without loss of generality, we may assume that

$$\pi(x_1) = L(\{e_1, e_2, e_3, \cdots, e_k\}),$$

and

$$\pi(x_2) = L(\{e_2, e_3, e_4, \cdots, e_{k+1}\}).$$

Now assume that there is some x_j, say x_3, such that

$$\pi(x_1) \cap \pi(x_2) \subset \pi(x_3).$$

Then we may assume that $\pi(x_3) = L(\{e_2, e_3, e_4, \cdots, e_k, e_{k+2}\})$. Now assume that there is some x_i, such that $\pi(x_i)$ does not contain $\pi(x_1) \cap \pi(x_2)$. Since

$$\dim(\pi(x_i) \cap \pi(x_1)) = \dim(\pi(x_i) \cap \pi(x_2)) = k-1,$$

we can choose a spanning set $\{u_1,u_2,u_3,\cdots,u_k\}$ for $\pi(x_i)$ such that $u_i \in \pi(x_1)$, for $i = 1,2,3,\cdots,k-1$, and $u_k \in \pi(x_2)$. Hence

$$\pi(x_i) \subset L(\{e_1,e_2,e_3,\cdots,e_{k+1}\}),$$

and so

$$\dim(\pi(x_i) \cap \pi(x_3)) < k-1.$$

But this contradicts the fact that $\dim(\pi(x_i) \cap \pi(x_3)) = k-1$. So

$$L(\{e_2,e_3,e_4,\cdots,e_k\}) \subset \pi(x_i),$$

and hence

$$\dim \bigcap_{i=1}^{m} \pi(x_i) = k-1.$$

Thus far, we have shown that if any three of the spaces $\pi(x_1),\pi(x_2),\pi(x_3),\cdots,\pi(x_m)$ intersect in a $k-1$ space, then they all intersect in a $k-1$ space. Now assume that no three of these spaces intersect in a $k-1$ space. Hence, for $i \neq 1,2$, $\pi(x_i)$ does not contain $\pi(x_1) \cap \pi(x_2)$. So, as before, $\pi(x_i) \subset L(\{e_1,e_2,e_3,\cdots,e_{k+1}\})$, and so

$$\dim L(\{\pi(x_i)\colon 1 \leqq i \leqq m\}) = k+1.$$

5. **Proof of Lemma 3.5.** The proof is in two parts.

PART 1. We first prove that, given the assumptions of the lemma, there is a set $\{x_1,x_2,x_3,\cdots,x_n\} \subset E_n$, such that

$$\pi(A_p) = L(\{x_{p_1},x_{p_2},x_{p_3},\cdots,x_{p_k}\}) \tag{1}$$

for any $p \in P(n,k)$. The proof is by induction on the number of vectors which can be found satisfying (1). First note that the assumption that for any $q \in P(n,k-1)$, the dimension of the intersection of the spaces $\pi(A_p)$ for $q \subset p \in P(n,k)$ is $k-1$, implies that to each $q \in P(n,k-1)$ there is assigned in a one-to-one manner, a $k-1$ space $S(q)$ of E_n, such that

$$S(q) = \pi(A_p) \cap \pi(A_r),$$

for any $p \in P(n,k)$, and $r \in P(n,k)$, such that $p \neq r$, and $q \subset p \cap r$. Obviously, there is a set $\{x_1, x_2, x_3,\cdots,x_k\} \subset E_n$ such that if $p = \{1,2,3,\cdots,k\}$, then (1) is true. So, assume that there exists a set $\{x_1,x_2,x_3,\cdots,x_t\} \subset E_n$, for some integer t, $k \leqq t \leqq n-1$, such that (1) holds for any $p \in P(t,k)$. Let $p = \{1,2,3,\cdots,k-1,t+1\}$. Then there exists an $x_{t+1} \in E_n$ such that (1) holds for this p. Let q be an element of $P(t,k-1)$, so ordered that if $q_s \in p$, then $q_s = s$. Let $\bar{p} = q \cup \{t+1\}$. We wish to show that (1) holds for $\bar{p}$. We now define a family of elements of $P(n,k)$ as follows:

$$p(0) = p,$$

$$p(j) = (p(j-1) - \{j\}) \cup \{q_j\},$$

for $j = 1, 2, 3, \cdots, k-1$. We will show by induction on j, that (1) holds for each $p(j)$. This will complete the induction on t, since $\bar{p} = p(k-1)$. Obviously, (1) is true if $j = 0$. Assume that, for some j, $0 \leqq j < k-1$, (1) holds for $p(j)$. If $q_{j+1} = j + 1$, then (1) holds for $p(j + 1)$. So assume that $q_{j+1} \notin p$. We also assume that $q_{j+1} \neq k$. Let

$$p' = (p(j + 1) - \{t + 1\}) \cup \{k\},$$

$$r = (p(j + 1) - \{t + 1\}) \cup \{j + 1\},$$

$$Z(i) = L(\{x_1, x_2, x_3, \cdots, x_i\}),$$

for i equal t or $t + 1$. Then

$$\pi(A_{p(j+1)}) \cap Z(t) = \pi(A_{p'}) \cap \pi(A_{p(j+1)}) = S(p' \cap p(j + 1))$$
$$= \pi(A_{p'}) \cap \pi(A_r) = L(\{x_{p(j+1)_1}, x_{p(j+1)_2}, x_{p(j+1)_3}, \cdots, x_{p(j+1)_{k-1}}\}),$$

and

$$\pi(A_{p(j)}) \cap \pi(A_{p(j+1)}) = S(p(j) \cap p(j + 1)).$$

Since $p' \cap p(j + 1) \neq p(j) \cap p(j + 1)$,

$$\dim(\pi(A_{p(j+1)}) \cap \pi(A_{p(j)}) \cap Z(t)) < k-1.$$

Also, since $Z(t + 1)$ is spanned by $\pi(A_{p(j)}) \cup Z(t)$,

$$\dim(\pi(A_{p(j+1)}) \cap Z(t + 1)) = k,$$

and hence

$$\pi(A_{p(j+1)}) \subset Z(t + 1).$$

Therefore, (1) holds for $p(j + 1)$. If $q_{j+1} = k$, interchange k and $j + 1$ in the argument above. This completes the proof of Part 1.

PART 2. As a consequence of Part 1, there is an $H \in \mathscr{A}_n$ such that AH^k is diagonal. Hence we can assume that A is diagonal.

$$A = \operatorname{diag}(a_p), \qquad \text{for } p \in P(n, k).$$

Now select any two integers g and h, such that $1 \leqq g$, $h \leqq n$, and $g \neq h$. Let q and r be two elements of $P(n, k-1)$, neither of which contains g or h. Let

$$p = q \cup \{g\},$$

$$p' = q \cup \{h\},$$

$$\bar{p} = r \cup \{g\},$$

and

$$\bar{p}' = r \cup \{h\}.$$

We want to show that

$$a_p a_{\bar{p}} = a_{p'} a_{\bar{p}'}.$$

As in Part 1, we construct two families of elements of $P(n,k)$.

$$p(0) = p,$$

$$p(j) = (p(j-1) - \{q_j\}) \cup \{r_j\},$$

for $j = 1,2,3,\cdots,k-1$ and

$$p'(0) = p',$$

$$p'(j) = (p'(j-1) - \{p_j'\}) \cup \{r_j\},$$

for $j = 1,2,3,\cdots,k-1$. Here we regard the $p(j)$ and $p'(j)$ as so ordered that g or h is always the last element. It suffices to prove that

$$a_{p(j-1)} a_{p'(j)} = a_{p(j)} a_{p'(j-1)} \tag{2}$$

for $j = 1,2,3,\cdots,k-1$. Let $y = e_{p(j-1)} + e_{p(j)} + e_{p'(j)} + e_{p'(j-1)}$. Then $y \in \psi(k,n)$. Therefore $Ay \in \psi(k,n)$. Thus Ay satisfies the Plucker identities, one of which may be written as (2), since only these four components of Ay are not zero. Now let

$$b(g,h) = a_p / a_{p'}.$$

Then $b(g,h)$ is independent of q, and for any three integers g, h, and s, $1 \leqq g,h,s \leqq n$,

$$b(g,s) = b(g,h)\, b(h,s).$$

Therefore, for $r \in P(n,k)$, and $r' = \{1,2,3,\cdots,k\}$,

$$a_r = \prod_{i=1}^{k} b(p_i, i)\, a_{r'},$$

where $\prod$ here indicates product. So if $B \in \mathscr{A}_n$

$$B = \operatorname{diag}(b(1,1), b(2,1), b(3,1), \cdots, b(n,1)),$$

and

$$\lambda = \left(\prod_{i=1}^{k} b(1,i)\right) a_{r'},$$

then

$$A = \lambda I B^k.$$

This completes the proof of Lemma 3.5.

6. **The orthogonal group.** In this section we let F be the field of real numbers. For m a positive integer, let $\cdot$ denote the usual inner product of E_m, and $|v|$ the usual norm. For $A \in \mathscr{A}_m$, let $A^{(i)}$ denote the ith row vector of the matrix of A.

6.1. LEMMA. *For m and A as above, if there exists a set $T \subset E_m$ such that*
1. *$e_i \in T$ for all integers i, $1 \leqq i \leqq m$,*
2. *$A^{(i)} \in T$ for all integers i, $1 \leqq i \leqq m$,*
3. *for all $v \in T$, $Av \in T$, and $A^{-1}v \in T$,*
4. *for all $v \in T$, $|Av| = |v|$,*

then A is orthonormal.

Proof. Since, for $v \in T$, $A^{-1}v \in T$, we have that

$$|v| = |AA^{-1}(v)| = |A^{-1}(v)|.$$

Now let $x_i = A^{-1}e_i$ for any integer i, $1 \leqq i \leqq m$. Then $|x_i| = 1$, and $Ax_i = e_i$. Hence $A^{(i)} \cdot x_i = 1$, and thus $|A^{(i)}| \geqq 1$. But

$$|AA^{(i)}|^2 = \sum_{j=1}^{m} (A^{(j)} \cdot A^{(i)})^2 = A^{(i)} \cdot A^{(i)}.$$

So

$$\sum_{j=1, j \neq i}^{m} \left(A^{(j)} \cdot A^{(i)} \right)^2 = A^{(i)} \cdot A^{(i)}(1 - A^{(i)} \cdot A^{(i)}).$$

Hence $|A^{(i)}| \leqq 1$. Thus, for any integers i and j, $1 \leqq i, j \leqq m$, $i \neq j$, $|A^{(i)}| = 1$, and $A^{(i)} \cdot A^{(j)} = 0$. Hence A is orthonormal.

6.2. THEOREM. *Let $A \in G$ such that for all $v \in \psi(k, n)$, $|Av| = |v|$. Then A is orthonormal, and there exist $B \in \mathscr{A}_n$, B orthonormal, and $C \in C_N$, $C^2 = I$, such that either $A = CB^k$, or $A = CJB^k$.*

Proof. This follows immediately from the previous lemma.

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